Dr. Ranjan Kumar
Assistant Professor, Dept. of Physics
U.R. College, Rosera, Samastipur
A constituent unit of L.N. Mithila University, Darbhanga.

Sem : IV (Session: 2023-27)

Subject : Electrodynamics

Unit : Maxwell's Equations

Topic : Maxwell's equation in differential form

Differential form :

(i) $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ (Gauss's Law)

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere's Law with Maxwell's correction)

* Force law - including electric & magnetic field

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$

In Physical terms (significance) :→

(i) $\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$ (ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ (iv) $\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

Maxwell's equation in free space :→

$\rho = 0$ & $\vec{J} = 0$

$$\Rightarrow \boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = 0 & \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array}}$$

Maxwell's Equation in Matter : →

In materials polarization of electric and magnetic field occurs.

For inside polarized matter, there will be accumulation of bound charge and current.

The electric polarization '$\vec{P}$' produces a bound charge density. i.e,

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

Similarly, A magnetic polarization / magnetisation '$\vec{M}$' produces a bound current. i.e,

$$J_b = \vec{\nabla} \times \vec{M}$$

→ Hence, any change in the electric polarisation would involves a flow of bound charge; i.e., $\vec{J_P}$, which must be included in the total current.

i.e, Mathematically :→

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial \vec{P}}{\partial t} da_\perp$$

$$\Rightarrow \boxed{\vec{J_P} = \frac{\partial \vec{P}}{\partial t}} \rightarrow \text{Polarisation current}$$

Difference :→

J_b : (bound current) Current associated with the magnetisation of the material and involves the spin and orbital motion of electrons.

J_P : (Polarisation Current) Polarisation current is the result of the linear motion of charge when the electric polarisation changes.

Continuity equation with J_P :→

$$\vec{\nabla} \cdot \vec{J_P} = \nabla \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{P}) = -\frac{\partial \rho_b}{\partial t}$$

The bound current varies in responce to the change in M.

$$\boxed{\vec{J}_b = \vec{\nabla} \times \vec{M}}$$

Hence, the total charge density can be seperated in two parts! free and bound.

i.e, $\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho = \frac{1}{\epsilon_0}\left(\rho_E - \vec{\nabla}\cdot\vec{P}\right)$$

$$\Rightarrow \quad \vec{\nabla}\cdot\left(\epsilon_0\vec{E} + \vec{P}\right) = \frac{1}{\epsilon_0}\rho_f$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{D} = \frac{1}{\epsilon_0}\rho_f} \quad \text{with} \quad \boxed{D = \epsilon_0\vec{E} + \vec{P}}$$

Also, the current density can be seperated in three parts: $\Rightarrow$

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = J_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\left(J_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} - \mu_0\left(\vec{\nabla}\times\vec{M}\right) = \mu_0 J_f + \mu_0\frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right)$$

$$\Rightarrow \quad \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = J_f + \frac{\partial\vec{D}}{\partial t}$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \times \vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}} \quad \text{with} \quad \boxed{\begin{aligned} \vec{H} &= \frac{1}{\mu_0}\vec{B} - \vec{M} \\ \& \quad \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \end{aligned}}$$

$\Rightarrow$ Maxwell's Equation in matter in terms of free charge and free current:

$$\boxed{\begin{aligned} &(i) \quad \vec{\nabla} \cdot \vec{D} = \rho_f && (ii) \quad \vec{\nabla} \cdot \vec{B} = 0 \\ &(iii) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} && (iv) \quad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

Relation b/w $\vec{D}, \vec{B}, \vec{E}$ & $\vec{H}$:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \qquad \vec{M} = \chi_m \vec{H}$$

$$\vec{D} = \epsilon \vec{E} \qquad \vec{H} = \frac{1}{\mu} \vec{B}$$

$$\epsilon = \epsilon_0 (1 + \chi_e) \quad \& \quad \mu = \mu_0 (1 + \chi_m)$$

Displacement current: $\rightarrow$

$$J_d = \frac{\partial \vec{D}}{\partial t}$$